Guaranteed Rate of Streaming Erasure Codes over Multi-Link Multi-hop Network

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***Abstract*—We study the problem of transmitting a sequence of messages (streaming messages) through a multi-link, multi-hop packet erasure network. Each message must be reconstructed in-order and under a strict delay constraint. Special cases of our setting with a single link on each hop have been studied recently - the case of a single relay-node, is studied in Fong et al [1]; the case of multiple relays, is studied in Domanovitz et al [2]. As our main result, we propose an achievable rate expression that reduces to previously known results when specialized to their respective settings. Our proposed scheme is based on the idea of concatenating single-link codes from [2] in a judicious manner to achieve the required delay constraints. We propose a systematic approach based on convex optimization to maximize the achievable rate in our framework.**

I. Introduction

Real-time interactive video streaming is becoming an integral part of people's lives throughout the world. The portion of the Internet traffic consumed by applications such as remote learning, remote working, cloud-based augmented reality and cloud-based multi-person games (all very sensitive to the latency of the network) is expected to grow significantly in the upcoming years [3].

The latency experienced by users is impacted by the propagation delay, the processing delay and by errors, which we model as packet erasures. There are two main mechanisms to handle packet erasures. Automatic repeat request (ARQ) is one popular method in which the receiver acknowledges to the transmitter which packets arrived and which did not, and the transmitter re-sends the erased packets again. See, e.g. [4]–[6].

While many works focused on improving the efficiency of ARQ, when considering strict low-latency constraints, none can overcome its basic requirement, which is that the overall latency will be higher than a three-time one-way trip delay. Another mechanism to handle erasures is forward error correction (FEC). A plurality of works analyzed the guaranteed rate for streaming codes while assuming a maximal number of erasures (either a burst or arbitrary) while assuming a single link between the source and the destination. See e.g., [7]–[11].

In [1], the analysis of streaming codes was extended to a three-node network with a single link on each hop and a coding scheme coined symbol-wise decode and forward (SWDF) was shown to achieve capacity when the maximal number of erasures occurs in the first link. In [2], the network model was further extended to multiple nodes with a single link in each hop and a different coding scheme named state-dependent symbol-wise decode and forward (SD-SWDF) was shown to achieve capacity when the maximal number of erasures occurs in the first link.

Even though in many cases there are multiple links connecting each node, basic routing protocols select only a single forwarding path for the traffic between each source-destination pair. It is well recognized that using multiple paths between the source and destination can improve throughput and/or robustness of transmission. See e.g., [12], [13].

The utilization of multiple paths for improving throughput was discussed in a plurality of papers [14], [15]. Utilization of multiple paths for low latency communication was discussed [16] where reducing the average delay was the focus. The combination of multiple paths and forward error correction was discussed in [17], but no guaranteed performance was derived.

In order to analyze the guaranteed rate of streaming erasure codes over the relayed network, we analyze an adversarial channel model. In this setting, the number of erasures per link is (globally) upper bounded. Our proposed scheme immediately extends to a more general model where the number of erasures in each sliding window is bounded. This model provides a tractable approximation to any statistical model and leads to an insightful analysis and a non-trivial coding scheme. Further, in many streaming applications, the important figure of merit is the worst-case guarantee (rather than, for example, the average performance). Hence, using the adversarial model provides useful insights on the trade-off between the worst-case erasure event the coding scheme needs to be prepared for and the guaranteed rate it can provide.

II. Relayed Network Model

A source node wants to send a sequence of messages $\{\mathbf{s}_t\}_{t=0}^{\infty}$ to a destination node with the help of H middle nodes $r_1, \ldots, r_H$ where there are L_j links between node r_{j-1} and r_j (which we denote that the jth hop). We assume links

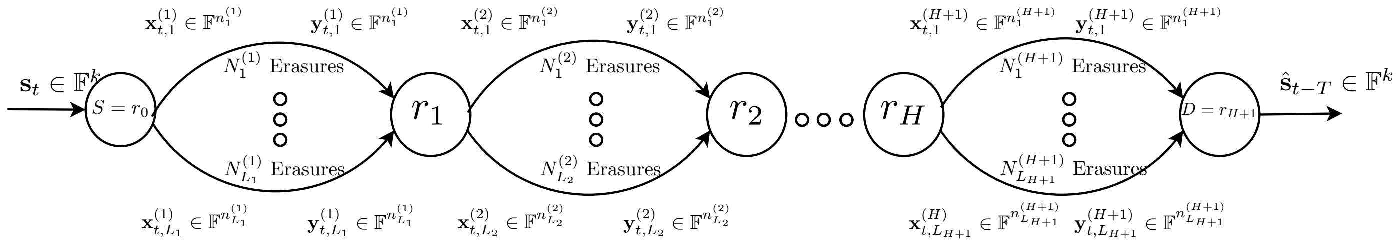

Fig. 1: Symbols generated in the $H+2$-node relay network at time i.

exists only between node j to node $j+1$ where each message consists of $k \geq 0$ symbols. We assume each link i in the jth hop introduces at most $N_i^{(j)}$ erasures. Every source message has to be recovered perfectly at the destination within a delay constraint of T time slots.

We denote $\mathbf{N}^{(j)} = [N_1^{(j)}, \ldots, N_{L_j}^{(j)}]$, and for simplicity, we denote the set $\mathbb{F}_e^n \triangleq \mathbb{F}^n \cup \{*\}$ (where $*$ indicates an erasure). This network is depicted in Fig. 1 and formalized below.

Definition 1. *Let $\mathbf{n}^{(j)} = [n_1^{(j)}, \ldots, n_{L_j}^{(j)}]$. An $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming code consists of the following:*

1) *A sequence of source messages $\{\mathbf{s}_t\}_{t=0}^{\infty}$ where $\mathbf{s}_t \in \mathbb{F}^k$.*
2) *A list of L_1 encoding functions $f_{t,i}^{(1)} : \underbrace{\mathbb{F}^k \times \ldots \times \mathbb{F}^k}_{t+1 \text{ times}} \to \mathbb{F}^{n_i^{(1)}}$, $i \in \{1, \ldots, L_1\}$ used by the source at time t to generate $\mathbf{x}_{t,i}^{(1)} = f_{t,i}^{(1)}(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_t)$.*
3) *A list of L_j relaying functions for each node $j \in [1, \ldots, H]$,*

$$f_{t,i}^{(j+1)} : \underbrace{\mathbb{F}_e^{n_1^{(j)}} \times \ldots \times \mathbb{F}_e^{n_1^{(j)}}}_{t+1 \text{ times}} \ldots \underbrace{\mathbb{F}_e^{n_{L_j}^{(j)}} \times \ldots \times \mathbb{F}_e^{n_{L_j}^{(j)}}}_{t+1 \text{ times}} \to \mathbb{F}^{n_i^{(j+1)}},$$

$i \in \{1, \ldots, L_j\}$ used at the jth node at time t to generate

$$\mathbf{x}_{t,i}^{(j+1)} = f_{t,i}^{(j+1)} \left(\{\mathbf{y}_{0,l}^{(j)}\}_{l=1}^{L_j}, \ldots, \{\mathbf{y}_{t,l}^{(j)}\}_{l=1}^{L_j} \right).$$

4) *A decoding function*

$$\varphi_{t+T} : \underbrace{\mathbb{F}_e^{n_1^{(H+1)}} \times \ldots \times \mathbb{F}_e^{n_1^{(H+1)}}}_{T+t+1 \text{ times}} \ldots \underbrace{\mathbb{F}_e^{n_{L_{H+1}}^{(H+1)}} \times \ldots \times \mathbb{F}_e^{n_{L_{H+1}}^{(H+1)}}}_{T+t+1 \text{ times}} \to \mathbb{F}^k.$$

is used by the destination (node r_{H+1}) at time $t+T$ to estimate $\mathbf{s_t}$ according to $\hat{\mathbf{s}}_t = \varphi_{t+T}(\{\mathbf{y}_{0,l}^{(H+1)}\}_{l=1}^{L_{H+1}}, \ldots, \{\mathbf{y}_{t+T,l}^{(H+1)}\}_{l=1}^{L_{H+1}})$.

Definition 2. *An erasure sequence is a binary sequence denoted by $e^{\infty} \triangleq \{e_t\}_{t=0}^{\infty}$, where $e_t = \mathbf{1}\{$an erasure occurs at time $t\}$.*

An N-erasure sequence is an erasure sequence e^{∞} that satisfies $\sum_{t=0}^{\infty} e_t^{\infty} = N$. In other words, an N-erasure sequence specifies N arbitrary erasures on the discrete timeline. The set of N-erasure sequences is denoted by Ω_N.

Definition 3. *The mapping $g_n : \mathbb{F}^n \times \{0,1\} \to \mathbb{F}_e^n$ of an erasure channel is defined as $g_n(\mathbf{x}, e) = \begin{cases} \mathbf{x} & \text{if } e = 0 \\ * & \text{if } e = 1. \end{cases}$ For any erasure sequence e^{∞} and any $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming code, the following input-output relation holds for the ith link in the jth hop ($j \in \{1, \ldots, H+1\}$), for each $t \in \mathbb{Z}_+$ $\mathbf{y}_{t,i}^{(j)} = g_{n_i^{(j)}}(\mathbf{x}_{t,i}^{(j)}, e_{t,i}^{(j)})$, where $e_{t,i}^{(j)} \in \Omega_{N_i^{(j)}}$, $i \in \{1, \ldots, L_j\}$.*

Definition 4. *An $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming code is said to be $(\mathbf{N}^{(1)}, \ldots \mathbf{N}^{(H+1)})$-achievable if, for any $e_{t,i}^{(j)} \in \Omega_{N_i^{(j)}}$, for all $j \in \{1, \ldots, H+1\}$, for all $i \in \{1, \ldots, L_j\}$, for all $t \in \mathbb{Z}_+$ and all $\mathbf{s}_t \in \mathbb{F}^k$, we have $\hat{\mathbf{s}}_t = \mathbf{s}_t$.*

Definition 5. *The rate of an $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming code is $\frac{k}{\max\left(\max(\mathbf{n}^{(1)}), \ldots \max(\mathbf{n}^{(H+1)})\right)}$.*

Remark 1. *Definition 5 suggests that the rate of a streaming code over multi-link multi-hop network can be greater than 1.*

A. Known results

Three-node network with a single link between the nodes ($H = 1,\ L_1 = L_2 = 1$):
In [1], it was shown that a coding scheme coined SWDF is an $\left(T - N_1^{(2)} + 1, T - N_1^{(1)} + 1, T - N_1^{(1)} - N_1^{(2)} + 1, T\right)_{\mathbb{F}}$ streaming code which is $(N_1^{(1)}, N_1^{(2)})$-achievable. It was further shown that when $N_1 \geq N_2$ this scheme achieves the upper bound thus capacity is established.

In high-level, this scheme splits the information message into symbols, encode each symbol such that it is guaranteed that it will be available at the relay with a different delay (using diagonally interleaved block codes). The relay then transmits the recovered symbols such that the overall delay constraint is met. For further details, see [1].

For example, assume that the source wishes to transmit two bits $[a_i, b_i]$ in every channel use over a three-node network with $N_1^{(1)} = N_1^{(2)} = 1$ with an overall delay of $T = 3$. The (capacity achieving) SWDF transmission is given in Table I below. Noting that it is guaranteed that b_i is available at the relay at time $i+1$ and a_i is available at the relay at time $i+2$ (for any single erasure in the link between the source and relay), it can be seen that for any single erasure in the link between the relay and destination, $[a_i, b_i]$ is guaranteed to be recovered at the destination at time $i+3$.

	Time	$i-1$	i	$i+1$	$i+2$	$i+3$
Source	a_i	a_{i-1}	a_i	a_{i+1}	a_{i+2}	a_{i+3}
	b_i	b_{i-1}	b_i	b_{i+1}	b_{i+2}	b_{i+3}
	$X_i = a_{i-2} + b_{i-1}$	X_{i-1}	X_i	X_{i+1}	X_{i+2}	X_{i+3}
Relay	b_{i-1}	b_{i-2}	b_{i-1}	b_i	b_{i+1}	b_{i+2}
	a_{i-2}	a_{i-3}	a_{i-2}	a_{i-1}	a_i	a_{i+1}
	$Y_i = b_{i-3} + a_{i-3}$	Y_{i-1}	Y_i	Y_{i+1}	Y_{i+2}	Y_{i+3}
Dest.	a_{i-3}	a_{i-4}	a_{i-3}	a_{i-2}	a_{i-1}	a_i
	b_{i-3}	b_{i-4}	b_{i-3}	b_{i-2}	b_{i-1}	b_i

TABLE I: SWDF at rate $2/3$ used over a network with $N_1^{(1)} = N_1^{(2)} = 1, T = 3$. Symbols marked with the frame type belong to the same block code.

Multi-node network with a single link between the nodes $(H = h,\ L_j = 1\ \forall j \in \{1, \dots, h+1\})$: Denoting $n_{\max} \triangleq \max_{j \in \{1,\dots,h+1\}} \left(T - \sum_{l=1, l\neq j}^{h+1} N_1^{(l)} + 1\right)$ and

$$\mathrm{OH} \triangleq \frac{n_{\max} \lceil \log(n_{\max}) \rceil}{\log(|\mathbb{F}|)}, \tag{1}$$

in [2], it was shown that when $T \geq \sum_{l=1}^{h+1} N_1^{(l)}$ SD-SWDF is an $\left(n^{(1)}_{\mathrm{SD-SWDF}}, \dots, n^{(H+1)}_{\mathrm{SD-SWDF}}, T - \sum_{l=1}^{h+1} N_1^{(l)} + 1, T\right)_{\mathbb{F}}$ streaming code which is $\left(N_1^{(1)}, \dots, N_1^{(h+1)}\right)$-achievable where

$$n^{(j)}_{\mathrm{SD-SWDF}} \triangleq T - \sum_{l=1, l \neq j}^{h+1} N_1^{(l)} + 1 + \mathrm{OH}. \tag{2}$$

In SD-SWDF, the transmission at each relay depends on the erasure pattern of the previous nodes (hence an additional header is added to allow nodes to decode symbols arriving in different order). It was further shown that when $N_1^{(1)} \geq N_1^{(l)}$, $\forall\, l > 1$ SD-SWDF approaches the upper bound (derived in [2]) when $|\mathbb{F}| \to \infty$ thus establishing capacity for this case. For further details, see [2].

III. Main Results

We present a coding scheme coined concatenated[1] SD-SWDF. While the suggested scheme is a straightforward extension of achievable schemes for a single-link multi-hop network, it nevertheless demonstrates the potential benefit of utilizing all links (compared to choosing a single path). We show that

Lemma 1. *Denoting with* $\bar{N}^{(j)} \triangleq \frac{1}{L_j} \sum_{i=1}^{L_j} N_i^{(j)}$, *when* $T \geq \sum_{l=1}^{H+1} \max(\mathbf{N}^{(l)})$, *the following rate is guaranteed*

$$R = \min_j \left(\frac{T + 1 - \sum_{l=1}^{H+1} \bar{N}^{(l)}}{\frac{1}{L_j}\left(T + 1 - \sum_{l \neq j} \bar{N}^{(l)} + \mathrm{OH}\right)} \right), \tag{3}$$

where OH *is defined in* (1).

[1]Concatenated codes have different meaning in the context of channel coding. In our context, concatenation is similar to string concatenation.

We then provide an optimization method that is used to further improve the achievable rate and provides non-trivial insights.

Remark 2. *The suggested achievable scheme (both with and without the optimization) holds for any* $T > \sum_{l=1}^{H+1} \min(\mathbf{N}^{(l)})$ *albeit, the overall rate can not be expressed in a compact form.*

IV. Achievable Scheme - concatenated SD-SWDF

Definition 6. *A concatenation of an* $(\mathbf{n}'^{(1)}, \mathbf{n}'^{(2)}, \dots, \mathbf{n}'^{(H+1)}, k', T)_{\mathbb{F}}$ *streaming code with an* $(\mathbf{n}''^{(1)}, \mathbf{n}''^{(2)}, \dots, \mathbf{n}''^{(H+1)}, k'', T)_{\mathbb{F}}$ *streaming code is an* $(\mathbf{n}'^{(1)} + \mathbf{n}''^{(1)}, \mathbf{n}'^{(2)} + \mathbf{n}''^{(2)}, \dots, \mathbf{n}'^{(H+1)} + \mathbf{n}''^{(H+1)}, k' + k'', T)_{\mathbb{F}}$ *streaming code with the following properties*

- *Let* $\mathbf{s}_t = [\mathbf{s'}_t\ \mathbf{s''}_t]^T$ *be the input to the concatenated code where* $s'_t \in \mathbb{F}^{k'}$ *and* $s''_t \in \mathbb{F}^{k''}$.
- *Let* $\{f_{t,i}^{(1)'}\}$ *and* $\{f_{t,i}^{(1)''}\}$ *be the encoding functions for node* S *of the first and second codes respectively. The encoding function of the concatenated code outputs* $\left\{\left[f_{t,i}^{(1)'}(\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_t), f_{t,i}^{(1)''}(\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_t)\right]^T\right\}_{i=1}^{L_1}$.
- *Let* $\{\mathbf{y}_{t,l}^{(j)}\}_{l=1}^{L_j}$ *denote the inputs to relay* r_j. *Let* $\{f_{t,i}^{(j+1)'}\}$ *and* $\{f_{t,i}^{(j+1)''}\}$ *be the relaying functions for node* r_j *of the first code and second code respectively. The relaying function of the concatenated code outputs* $\left\{\left[f_{t,i}^{(j+1)'}(\{\mathbf{y}_{t,l}^{(j)}\}_{l=1}^{L_j}), f_{t,i}^{(j+1)''}(\{\mathbf{y}_{t,l}^{(j)}\}_{l=1}^{L_j})\right]^T\right\}_{i=1}^{L_{j+1}}$.
- *Let* $\{\mathbf{y}_{t,l}^{(H+1)}\}_{l=1}^{L_{H+1}}$ *denote the input to the decoder of the concatenated code. Denote* $\{\varphi'_t\}$ *as the decoding functions of the first code and* $\{\varphi''_t\}$ *as the decoding functions of the second code. The output of the concatenated code is*

$$\hat{\mathbf{s}}_t = \left[\underbrace{\varphi'_{i+T}\left(\mathbf{y}_0^{(H+1)}, \dots, \mathbf{y}_{t+T}^{(H+1)}\right)}_{\hat{\mathbf{s}'}_t}, \underbrace{\varphi''_{i+T}\left(\mathbf{y}_0^{(H+1)}, \dots, \mathbf{y}_{t+T'}^{(H+1)}\right)}_{\hat{\mathbf{s}''}_t}\right]^T.$$

Corollary 1. *Following Definition 5, we note that the rate of the concatenated code is* $\frac{k' + k''}{\max_j(\mathbf{n}'^{(j)} + \mathbf{n}''^{(j)})}$.

We denote concatenation of the same code M times as transmitting this code with multiplicity M. Next, we formally define a path over the relayed network.

Definition 7. *In a relayed network with* $H+1$ *nodes with* L_j *links in the* j*th hop* $(1 \leq j \leq H+1)$, *the (unique)* m*th path from the source to destination is defined as the set of indices* $\{i_m^1, i_m^2, \dots, i_m^{H+1}\}$ *where* $1 \leq i_m^1 \leq L_1,\ 1 \leq i_m^2 \leq L_2,\ \dots\ 1 \leq i_m^{H+1} \leq L_{H+1}$, *i.e.,* i_m^j *is the index of the link used in the* j*th hop by path* m. *For* $m \neq m'$ *there exists* $1 \leq h \leq H+1$ *such that* $i_m^h \neq i_{m'}^h$.

The suggested coding scheme transmits a concatenation of multi-hop single-link SD-SWDF codes. Hence, we denote it as CSD-SWDF.[2] We note that the suggested scheme limits the relaying functions($f_{t,i}^{(j+1)}$ of Definition 1) to perform operations

[2]When a three-node multi-link network is considered, SWDF can be used.

only on the "path level" (i.e. without "mixing" information symbols from different paths). While this limitation to the relay operation may not be optimal, it allows us to bound the maximal size of the needed overhead and to derive closed-form expressions for the guaranteed achievable rate.

Transmitting SD-SWDF streaming code over the mth path is equivalent to transmitting

$$\Big([0,\dots,0,n^{(1)}_{i^1_m},0,\dots,0],\dots,[0,\dots,0,n^{(h+1)}_{i^{h+1}_m},0,\dots,0],\\ T+1-\sum_{l=1}^{h+1} N_{i^l_m},T\Big)_{\mathbb{F}}$$

streaming code where $n^{(j)}_{i^j_m} \triangleq T-\sum_{l=1,l\neq j}^{h+1} N^{(l)}_{i^l_m}+1+\mathrm{OH}$ is rewriting (2) using the indices which belong to path m defined in Definition 7.

Proof of Lemma 1. Since CSD-SWDF transmits simultaneously $\prod_s L_s$ codes over $\prod_s L_s$ different paths without interaction between the codes, and since each code used on the mth path is $(N^{(1)}_{i^1_m},\dots N^{(H+1)}_{i^{H+1}_m})$-achievable it follows that CSD-SWDF is $(\mathbf{N}^{(1)},\dots\mathbf{N}^{(H+1)})$-achievable.

From corollary 1 we have that the total number of information symbols transmitted by the suggested scheme is

$$\begin{aligned} k_{tot} &= \sum_m (T+1-\sum_{l=1}^{h+1} N_{i^l_m}) \\ &= \prod L_s\cdot(T+1)-\prod_{s\neq 1} L_s\cdot\left(N^{(1)}_1+\dots+N^{(1)}_{L_1}\right)\\ &-\dots-\prod_{s\neq H+1} L_s\cdot\left(N^{(H+1)}_1+\dots+N^{(H+1)}_{L_{H+1}}\right)\\ &= \prod L_s\cdot\left(T+1-\sum_{l=1}^{H+1}\bar{\mathbf{N}}^{(l)}\right). \end{aligned}$$

Noting that link i in hop j is part of $\prod_{s\neq j} L_s$ paths (in the other $\prod_s L_s-\prod_{s\neq j} L_s$ paths, other links from this hop are used. Hence, equivalently, it is assigned with packet size zero), we further have from Definition 6 that the size of the packet sent over link i in hop j is

$$\begin{aligned} n^{(j)}_i &= \sum_m n^{(j)}_{i^j_m} \\ &= \prod_{s\neq j} L_s\cdot(T+1+\mathrm{OH})-\sum_{u\neq j}(\prod_{s\neq u,j} L_s)\cdot\left(\sum_{v=1}^{L_u} N^{(u)}_v\right)\\ &= \prod_{s\neq j} L_s\cdot(T+1+\mathrm{OH})-\sum_{u\neq j}(\prod_{s\neq u,j} L_s)\cdot\left(L_u\bar{\mathbf{N}}^{(u)}\right)\\ &= \prod_{s\neq j} L_s\cdot\left(T+1+\mathrm{OH}-\sum_{u\neq j}\left(\bar{\mathbf{N}}^{(u)}\right)\right)\triangleq n^{(j)} \end{aligned} \tag{4}$$

Noting that the size of the packet sent over link i in hop j is the same for all L_j links we have from corollary 1 that

$$\begin{aligned} R &= \frac{k_{tot}}{\max_j n^{(j)}}\\ &= \frac{\prod L_s\cdot\left(T+1-\sum_{l=1}^{H+1}\bar{\mathbf{N}}^{(l)}\right)}{\max_j\left(\prod_{s\neq j} L_s\cdot\left(T+1+\mathrm{OH}-\sum_{u\neq j}\left(\bar{\mathbf{N}}^{(u)}\right)\right)\right)}. \end{aligned}$$

□

Time	i	$i+1$	$i+2$	$i+3$	$i+4$	$i+5$
$x^{(1)}_{i,1}[1]=a_i$	$\boxed{a_i}$	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}	a_{i+5}
$x^{(1)}_{i,1}[2]=X$	X	$\boxed{X}$	X	X	X	X
$x^{(1)}_{i,1}[3]=X$	X	X	$\boxed{X}$	X	X	X
$x^{(1)}_{i,1}[4]=X$	X	X	X	$\boxed{X}$	X	X
$x^{(1)}_{i,1}[5]=b_i$	$\boxed{b_i}$	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}	b_{i+5}
$x^{(1)}_{1}[6]=c_i$	c_i	$\boxed{c_{i+1}}$	c_{i+2}	c_{i+3}	c_{i+2}	c_{i+3}
$x^{(1)}_{i,1}[7]=X$	X	X	$\boxed{X}$	X	X	X
$x^{(1)}_{i,1}[8]=X$	X	X	X	$\boxed{X}$	X	X
$x^{(1)}_{i,1}[9]=X$	X	X	X	X	$\boxed{X}$	X

TABLE II: Transmission over the first link in the first hop, concatenating SWDF codes sent between the source and the relay over paths 1 and 2 (transmitting 3 bits of information using a packet of size 9 bits), X stands for a parity symbol. Symbols marked with the frame type belong to the same block code.

A. Example

Consider a network with three nodes with two links connecting each node, where $\mathbf{N}^{(1)}=[3\ 2]$, $\mathbf{N}^{(2)}=[2\ 1]$ and $T=5$ which is depicted in Figure 2. This network can be decomposed into four paths, as also depicted in Figure 2. We note that since in all these paths $N_1\geq N_2$ SWDF achieves the optimal guaranteed rate (per path). Transmitting only on the best path (which is path 4 in this example) results in $R=3/5$.

Another simple coding scheme is to transmit SWDF code over non-overlapping paths. In the network depicted in Figure 2 it amounts to transmitting over paths 1 and 4 or 2 and 3. Both options result in $R=4/5$.[3]

Applying the suggested scheme amounts to transmit on each path (single-path capacity-achieving) SWDF code and concatenate the codes transmitted over each link. The codes used are

$$\text{Path }1:([4,0],[3,0],1,5)_{\mathbb{F}};\ \text{Path }2:([5,0],[0,3],2,5)_{\mathbb{F}};$$
$$\text{Path }3:([0,4],[4,0],2,5)_{\mathbb{F}};\ \text{Path }4:([0,5],[0,4],3,5)_{\mathbb{F}},$$

which result in a $([9,9],[7,7]),8,5)_{\mathbb{F}}$ streaming code with $R=\frac{1+2+2+3}{\max(9,7)}=\frac{8}{9}$.

Table II describes the packets sent over the first link of the first hop (assuming the source wishes to send 9 bits $[a_i,b_i,c_i,A_i,B_i,C_i,D_i,E_i,F_i]$ at each time instance).

B. An improved coding scheme

The basic scheme transmits SD-SWDF code over each path. A natural extension to this scheme is to study transmitting SD-SWDF with different multiplicities on different paths.

We note that the improved scheme does not require additional headers when SD-SWDF code is used per path. When an erasure occurs, it'll occur simultaneously to all the codes transmitted over this link. Thus, a single header is needed (per path) to allow each node to perform its operations. Specifically,

[3]Definition 5 suggests that transmitting SWDF (or SD-SWDF) over all non-overlapping paths might not improve upon transmitting over the best path.

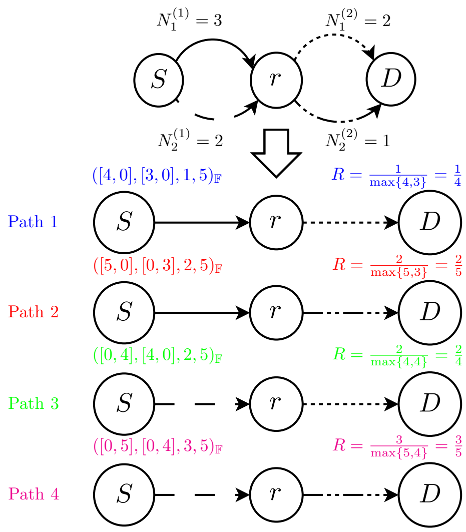

Fig. 2: Three-node network with two links between each node, $\mathbf{N}^{(1)} = [3\ 2]$, $\mathbf{N}^{(2)} = [2\ 1]$, $T = 5$, each link is marked with a unique line type, decomposed into four paths.

the number of the required headers is the same as in the basic scheme (one header per one path).

Denoting with c_m the multiplicity of the code used in the mth path, the total number of information symbols transmitted by the suggested scheme is

$$k_{tot} = \sum_m c_m (T + 1 - \sum_{l=1}^{h+1} N_{i_m^l}),$$

and the size of the packet sent over link i in hop j is

$$n_i^{(j)} = \sum_{i_m^j = i} c_m n_{i_{j,m}}^{(j)}.$$

We note that when different multiplicity per path is used, the size of the packet transmitted over different links which belong to the same hop is no longer the same (as was the case in (4)). Hence, denoting $\mathbf{n}^{(j)} = \left[n_1^{(j)}, \ldots, n_{L_j}^{(j)}\right]$, using Definition 5 the achievable rate of the optimized scheme is

$$R = \max_{c_m, 1 \le m \le \prod L_l} \frac{k_{\text{tot}}}{\max_j \left(\max(\mathbf{n}^{(j)})\right)}.$$

This can be converted to the following (convex) optimization problem

$$\max_{\tilde{c}_m} \frac{k_{\text{tot}}}{D} \quad \text{Where}\ :\ 1)\ 0 \le \tilde{c}_m \le 1 \quad 2) \sum_{m=1}^{\prod_l L_l} \tilde{c_m} = 1$$
$$3)\ D \ge n_i^{(j)}, \quad 1 \le j \le H+1,\ 1 \le i \le L_j,$$

which is tractable for some practical examples, as we demonstrate below. We note that this is a convex problem since both k_{tot} and D are positive (all $n_i^{(j)}$ are positive and there is at least one $n_i^{(j)} > 0$).

The amount of multiplicity to use per path is derived by taking $c_i = \lfloor \tilde{c}_i \cdot c \rfloor$ where c is a constant (same constant for all $\tilde{c}_i$) chosen to trade performance with overall packet size.

For the network depicted in Figure 2, not transmitting on path number 1, and transmitting on paths $2, 3, 4$ with multiplicities of $8, 5, 4$, i.e, the codes used are

$$\text{Path 1} : ([0,0],[0,0],0,5)_{\mathbb{F}}; \quad \text{Path 2} : ([40,0],[0,24],16,5)_{\mathbb{F}};$$
$$\text{Path 3} : ([0,20],[20,0],10,5)_{\mathbb{F}}; \quad \text{Path 4} : ([0,20],[0,16],12,5)_{\mathbb{F}},$$

which results in a $([40,40],[20,40],38,5)_{\mathbb{F}}$ streaming code with $R = \frac{16+10+12}{\max(40,20)} = 0.95$ (which improves upon $R = 8/9$ achieved with multiplicity 1 for all paths).

Remark 3. *Sometimes the optimization suggests not to transmit anything over some links. For example, in a three-node network with two links between each node,* $\mathbf{N}^{(1)} = [5\ 4]$, $\mathbf{N}^{(2)} = [2\ 1]$, $T = 7$, *the optimization results in not transmitting over the first link in the 2nd hop* ($N_1^{(2)} = 2$)*. Since concatenation results in a rate that is lower than each of the original rates, forcing transmission through paths that include this link will result in a degradation in the overall guaranteed rate. The average performance of the scheme can be further improved by using repetition on links with packet sizes smaller than the maximum (by gaining diversity). This requires a modification in the header to indicate which symbols arrived at each link. Hence, careful analysis of this scheme is left for further study.*

V. Numerical Results

Figure 3 depicts the ratio of the guaranteed rate achieved by the CSD-SWDF scheme and the guaranteed rate achieved by transmitting over a single (best) path. We further plot the ratio of using the optimized scheme described in Section IV-B. We plot the cumulative distribution function (CDF) of 1000 four-node relayed networks with two, three and four links between each node. The maximal number of erasures per link is randomly chosen (between 1 to 10) and the overall delay constraint is taken as $T = \sum_j \max(\mathbf{N}^j)$. The rate of the optimized scheme was found using CVXPY [18]. Using the suggested coding scheme results in a significant improvement compared to using a single path, and, as expected, the gain increases as the overall number of paths increases.

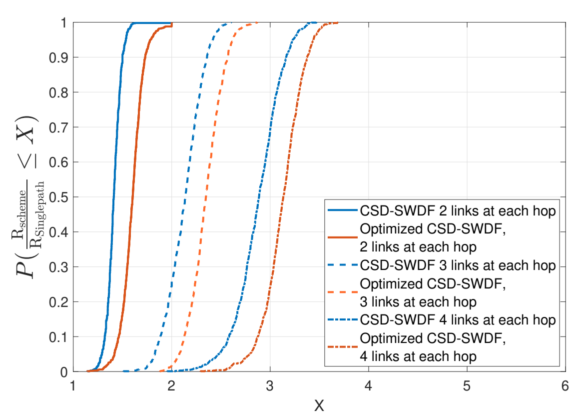

Fig. 3: The ratio between the guaranteed rates of CSD-SWDF and single (best) path transmission over four-node relayed network with various number of links between each node.

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